1 . A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is $355\,\text{cm}^3$

The metal used

- for the circular base and the curved side costs 0.04 pence/cm^2
- for the circular top costs 0.09 pence/cm^2

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{28.4}{r}$$

(4)

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4)

(c) Using $\frac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b).

(2)

(d) Hence find the minimum value of C, giving your answer to the nearest integer.

(2)

a) $\pi r^2 h = 355 \quad \therefore h = \frac{355}{\pi r^2}$ (1)

cost of base $= 0.04\,\pi r^2$

cost of side $= 0.04 \times 2\pi r h = 0.04 \times 2\pi r\left(\frac{355}{\pi r^2}\right) = \frac{28.4}{r}$

cost of top $= 0.09\,\pi r^2$

total cost, $C = 0.04\,\pi r^2 + \frac{28.4}{r} + 0.09\,\pi r^2$ (1)

$C = 0.13\,\pi r^2 + \frac{28.4}{r}$ (1)

b) C is minimum when $\frac{dC}{dr} = 0$

$$\frac{dC}{dr} = 0.26\pi r - \frac{28.4}{r^2} \quad (1)$$

$$\frac{dC}{dr} = 0, \quad 0.26\pi r - \frac{28.4}{r^2} = 0$$

$$r^3 - \frac{28.4}{0.26\pi} = 0$$

$$r^3 = \frac{28.4}{0.26\pi} \quad (1)$$

$$r = 3.26 \quad (3\text{ s.f.}) \quad (1)$$

c) $$\frac{d^2C}{dr^2} = 0.26\pi + \frac{56.8}{r^3} \quad (1)$$

when $r = 3.26$, $0.26\pi + \frac{56.8}{(3.26)^3}$

$= 2.45$ which is > 0. Hence, cost is minimised. (1)

d) when $r = 3.26$, $C = 0.13\pi(3.26)^2 + \frac{28.4}{3.26}$ (1)

$= 13$ (1)

$\therefore$ The minimum cost is 13p.

2. A curve has equation

$$y = \frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + 5$$

(a) Find $\frac{dy}{dx}$ writing your answer in simplest form.

(2)

(b) Hence find the range of values of x for which y is decreasing.

(4)

1 (a) $y = \frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + 5$ (1)

$\frac{dy}{dx} = 2x^2 - 7x - 4$ (1)

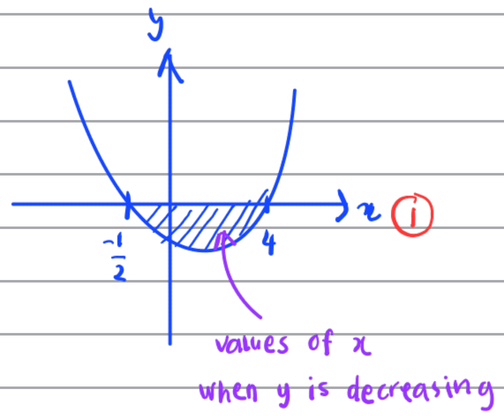
(b) $\frac{dy}{dx} = 0$

$2x^2 - 7x - 4 = 0$

$(2x+1)(x-4) = 0$ (1)

values of x : $-\frac{1}{2}$, 4 (1)

$-\frac{1}{2} < x < 4$ (1)



3. Given that

$$y = \frac{x-4}{2+\sqrt{x}} \qquad x > 0$$

show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

$$y = \frac{x-4}{2+x^{1/2}} \qquad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$\frac{dy}{dx} = \frac{(2+x^{1/2}) - \frac{1}{2}x^{-1/2}(x-4)}{(2+x^{1/2})^2} \text{ (1)} \text{ (1)} = \frac{2+x^{1/2} - \frac{1}{2}x^{1/2} + 2x^{-1/2}}{(2+x^{1/2})^2}$$

$$= \frac{2+\frac{1}{2}x^{1/2} + 2x^{-1/2}}{(2+x^{1/2})^2} \text{ (1)} \quad \times 2\sqrt{x}$$

$$= \frac{4x^{1/2} + x + 4}{2x^{1/2}(2+x^{1/2})^2} = \frac{\cancel{(2+x^{1/2})^2}}{2\sqrt{x}\,\cancel{(2+x^{1/2})^2}} = \frac{1}{2\sqrt{x}} \text{ (1)}$$

Alternative: $y = \dfrac{x-4}{2+\sqrt{x}} = \dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{2+\sqrt{x}} = \sqrt{x} - 2$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

4. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) (i) $y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$

$\dfrac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ (1)

(1)

(ii) $\dfrac{d^2y}{dx^2} = 60x^2 - 144x + 84$ (1)

b) (i) If C has a stationary point at $x=1$, then $\left.\dfrac{dy}{dx}\right|_{x=1} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 20(1)^3 - 72(1)^2 + 84(1) - 32$

$= 20 - 72 + 84 - 32 = 0$ ✓ (1)

so there is a stationary point at $x=1$ (1)

(ii) $\left.\dfrac{d^2y}{dx^2}\right|_{x=0.8} = 7.2 > 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=1.2} = -2.4 < 0$ (1)

Since there is a change in sign, $x=1$ is a point of inflection. (1)

5.

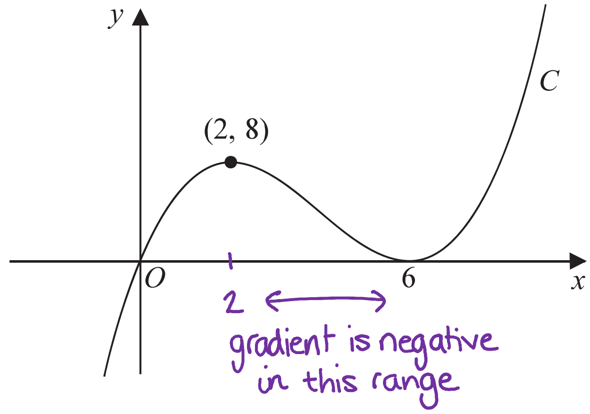


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ where $f(x)$ is a cubic expression in x.

The curve

- passes through the origin
- has a maximum turning point at $(2, 8)$
- has a minimum turning point at $(6, 0)$

(a) Write down the set of values of x for which

$$f'(x) < 0$$

(1)

The line with equation $y = k$, where k is a constant, intersects C at only one point.

(b) Find the set of values of k, giving your answer in set notation.

(2)

(c) Find the equation of C. You may leave your answer in factorised form.

(3)

(a) $2 < x < 6$ 1

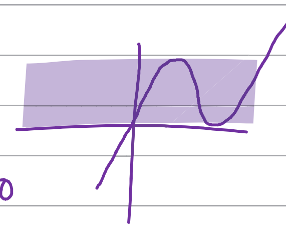
$f'(x) < 0$ means the gradient is negative.
Negative gradient = line going down. ↘

(b) $k > 8$ or $k < 0$ 1

$y=k$ is a horizontal line through the y-axis.

$\{k : k > 8\} \cup \{k : k < 0\}$ 1

has to be outside of shaded region to intersect only once. →



CHOOSE ONE OF THESE METHODS.

(c) Method 1 : Recognise curve has form $y = ax(x-6)^2$ ① states form of c

$(2,8) \rightarrow \quad 8 = 2a(2-6)^2$ ①

$8 = 32a$

$a = \frac{1}{4}$

$\therefore y = \frac{1}{4}x(x-6)^2$ ①

Method 2 : Solving Simultaneous Equations

$y = ax^3 + bx^2 + cx$ ← no +d because the curve goes through the origin.

when $x = 2$, $y = 8$:

$8 = a(2^3) + b(2^2) + c(2)$

① $4 = 4a + 2b + c$

when $x = 6$, $y = 0$:

$0 = a(6^3) + b(6^2) + c(6)$

② $0 = 216a + 36b + 6c$ ① for 2 sim. eq.

$f'(x) = 3ax^2 + 2bx + c$

when $x = 6$, $f'(x) = 0$: ← (6,0) is a turning point

$0 = 3a(6^2) + 2b(6) + c$

③ $0 = 108a + 12b + c$

Solve ①, ②, ③ simultaneously : ← use a calculator or solve by hand.

$4 = 4a + 2b + c$

$0 = 216a + 36b + 6c$

$0 = 108a + 12b + c$

$a = \frac{1}{4}$, $b = -3$, $c = 9$ ① for solving sim. eq.

$y = \frac{1}{4}x^3 - 3x^2 + 9x$ ①

6.

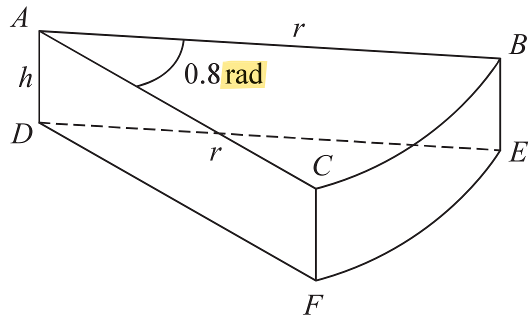


Figure 5

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD, CF and BE are perpendicular to faces ABC and DEF
- edges AD, CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

(a) show that the surface area of the toy, $S\text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)

(a) $\underbrace{\frac{1}{2} \times 0.8 \times r^2}_{\text{area of sector}} \times \underbrace{h}_{\text{height}} = \underbrace{240}_{\text{volume}}$ 1 $\qquad \frac{1}{2}\theta r^2 =$ area of sector (when θ is in radians)

$0.4r^2h = 240$ $\div 0.4$

$r^2h = 600$ $\div r^2$

$h = \frac{600}{r^2}$ 1

total surface area = 2 × area of sector face + 2 × area of sector length + area of arc

$S = 2\left(\frac{1}{2}\theta r^2\right) + 2(rh) + (r\theta \times h)$

$S = 0.8r^2 + 2rh + 0.8rh$

$S = 0.8r^2 + 2r\left(\frac{600}{r^2}\right) + 0.8r\left(\frac{600}{r^2}\right)$ (1) $h = \frac{600}{r^2}$

$S = 0.8r^2 + \frac{1200}{r} + \frac{480}{r}$

$S = 0.8r^2 + \frac{1680}{r}$ (1)

(b) $S = 0.8r^2 + 1680r^{-1}$ $\quad \frac{1}{x} = x^{-1}$

$\frac{dS}{dr} = 0.8 \times 2r^{2-1} + (-1) \times 1680r^{-1-1}$

$= 1.6r - 1680r^{-2}$ (2)

$0 = 1.6r - \frac{1680}{r^2}$ (1) ← $\frac{dS}{dr} = 0$ at stationary point

$1.6r = \frac{1680}{r^2}$ $\quad \times r^2$

$1.6r^3 = 1680$ $\quad \div 1.6$

$r^3 = 1050$ $\quad \sqrt[3]{\ }$

$r = \sqrt[3]{1050}$

$r = 10.16$ (1)

(c) $\frac{dS}{dr} = 1.6r - 1680r^{-2}$

$\frac{d^2S}{dr^2} = 1.6 \times 1r^{1-1} - (-2) \times 1680r^{-2-1}$

$= 1.6 + 3360r^{-3}$

$= 1.6 + \frac{3360}{r^3}$

when $r = 10.16$: ← from part (b), stationary point at $r = 10.16$

$1.6 + \frac{3360}{(10.16)^3} = 4.80$ (1)

$\frac{d^2S}{dr^2} > 0$ when $r = 10.16$ therefore this is a minimum value of S. (1)

7.

$$f(x) = x^3 + 2x^2 - 8x + 5$$

(a) Find $f''(x)$

(2)

(b) (i) Solve $f''(x) = 0$

(ii) Hence find the range of values of x for which $f(x)$ is concave.

(2)

a) $f(x) = x^3 + 2x^2 - 8x + 5$

$f'(x) = 3x^2 + 4x - 8$

$f''(x) = 6x + 4$ (1) (1)

b)(i) $6x + 4 = 0$

$6x = -4$

$x = -\frac{2}{3}$ (1)

(ii) concave when $f''(x) < 0$

$6x + 4 < 0$

$x < -\frac{2}{3}$ (1)

concave: "the rate of change of gradient is decreasing"